MEASURABLE RECURRENCE AND QUASI-INVARIANT MEASURES

BY

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ABSTRACT

We characterize the wandering sets of a Borel automorphism T as being precisely those sets which have measure zero for every non-atomic measure μ which is quasi-invariant and ergodic for T.

§1. Introduction

In ergodic theory one studies the recurrence properties of a transformation $T: X \to X$ when X is endowed with the structure of a measure space, while in topological dynamics one studies analogous questions when X is a topological space. It seems very natural to investigate properties of T when X is merely a measured space, i.e., equipped with a σ-algebra of subsets. Let $(X, \mathcal{B})$ be a standard Borel space, say X the unit interval and $\mathcal{B}$ the Borel subsets of X, and suppose that $T: X \to X$ is an automorphism of $(X, \mathcal{B})$, that is to say, T is one to one and onto and $T(\mathcal{B}) = \mathcal{B}$. For any $B \in \mathcal{B}$ the proof of the Poincaré recurrence theorem shows that there exists $B_0 \subset B$ such that $B_0 \in \mathcal{B}$ and

(i) all $x \in B_0$, $T^n x \in B_0$ for infinitely many positive values of n;

(ii) there exists a *wandering* set $W \in \mathcal{B}$ (i.e., $W \cap T^n W = \emptyset$ for all $n \neq 0$) and $B \setminus B_0 \subset \bigcup_{-\infty}^{\infty} T^n W$.

Indeed one can take for W the set of those $x \in B$ such that $T^n x \notin B$ all $n \geqq 1$ and put $B_0 = B \setminus \bigcup_{-\infty}^{\infty} T^n W$. It is straightforward to check that the collection $\mathcal{W}$ of sets E that lie in $\bigcup_{-\infty}^{\infty} T^n W$ for some wandering set $W \in \mathcal{B}$ form a σ-ideal, i.e., $\mathcal{W}$ is closed under countable unions and passage to subsets (in $\mathcal{B}$ of course). If one thinks of the sets in $\mathcal{W}$ as being trivial then we obtain the most trivial version of Poincaré's recurrence theorem (cf. the discussion in [2, ch. 17]) that, except for a trivial set (which may of course be all of B), all points of B return to B infinitely often. Our goal in this paper is to give another characterization of

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the σ-ideal $\mathcal{W}$ in terms of $\mathcal{M}$, the collection of non-atomic probability measures μ on $(X, \mathcal{B})$ which are quasi-invariant and ergodic for T. Letting $\mathcal{N}(\mu)$ denote the σ-ideal of null sets for μ, observe that $\mathcal{W} \subset \mathcal{N}(\mu)$ for any μ. The characterization we are after is

THEOREM. $\mathcal{W} = \bigcap_{\mu \in \mathcal{M}} \mathcal{N}(\mu)$.

A simple corollary of this characterization is the fact that if $\mathcal{W} \neq \mathcal{B}$, or equivalently $X \not\in \mathcal{W}$, then $\mathcal{M}$ is non-empty. In case X is given the structure of a complete separable metric space and T is assumed to be continuous then this corollary follows easily from the result of [1], and in a sense the construction given here is modelled after the one there. Here is another easy consequence (answering a question raised by T. Kamae): if T is an automorphism of a standard Borel space then $\mathcal{W}$ contains uncountable sets, in particular if $S : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ is the usual 2-shift $((Sx)_n = x_{n+1})$ and A is an uncountable S-invariant Borel subset of $\{0,1\}^{\mathbb{Z}}$ then A has an uncountable wandering set. We conclude this introduction by outlining our strategy for proving the main result. We start with any set $A \in \mathcal{B}$ but not in $\mathcal{W}$ and construct a subset $B \subset A$ and an identification of B with $\{0,1\}^{\mathbb{N}} = Y$ so that T_B, the transformation induced by T on B, i.e. $T_B x = T^n x$ where $n = \min\{n \geqq 1 : T^n x \in B\}$, becomes the odometer on Y. Recall that the odometer $D : Y \to Y$ is defined by $Dy = y + 1$, where the addition is that of the 2-adic integers with the usual identification of Y with the 2-adic integers. Having done this we are done, because we can put on (B, T_B) an abitrary non-atomic measure μ_B which is quasi-invariant and ergodic for (Y, D) and then in an obvious way extend μ_B to a $\mu \in \mathcal{M}$, and we have that $\mu(B) > 0$, and thus $B \not\in \mathcal{N}(\mu)$.

In §2 we prove a result about a certain game which will serve as a substitute for completeness in showing that the B we construct in §3 has the structure of Y. The fact that we assume $(X, \mathcal{B})$ to be a standard Borel space is used in an essential way in §2. Even if $\mathcal{B}$ is assumed countably generated $(X, \mathcal{B})$ may carry no countably additive measure so that $\mathcal{M}$ can be empty for all T, whereas $\mathcal{W}$ is certainly empty for the identity mapping, so that the result can fail dramatically for general measured spaces.

§2. The game and its winning strategy

In this section $\mathcal{E}$ can be any σ-ideal in $\mathcal{B}$, the Borel subsets of the unit interval X, and sets in $\mathcal{B} \setminus \mathcal{E}$ will be called *non-trivial*. The game we consider is played between two players, I, II, who, beginning with player I, alternate in choosing a non-trivial subset contained in the previous choice of the other player. Player I wins if no point of X is contained in all the choices while player II wins in the contrary case. In detail: player I selects some non-trivial A_0, player II selects some non-trivial $B_0 \subset A_0, \cdots$, player I selects some non-trivial $B_n \subset A_n, \cdots$, player I wins if $\bigcap_0^\infty B_n \neq \emptyset$. In selecting A_n or B_n the players retain complete information concerning the previous choices.

PROPOSITION 1. *In the game described above player II has a winning strategy.*

PROOF. We need an explicit representation of $\mathcal{B}$, and sets in $\mathcal{B}$. Let $\mathcal{B}_0$ consist of the closed subsets of X, the unit interval, $\mathcal{B}_1$ = countable unions of elements of $\mathcal{B}_0$, $\mathcal{B}_2$ = countable intersections of sets in $\mathcal{B}_1, \cdots, \mathcal{B}_\alpha = \bigcup_{\beta<\alpha} \mathcal{B}_\beta$ for a limit ordinal α, and for any ordinal α, define $\mathcal{B}_{\alpha+1}$ to be countable unions of elements of $\mathcal{B}_\alpha$, $\mathcal{B}_{\alpha+2}$ to be countable intersections of elements of $\mathcal{B}_{\alpha+1}$, etc., alternating unions and intersections. Since the complement of any element in $\mathcal{B}_0$ is in $\mathcal{B}_1$ (any open set in $[0,1]$ is an $\mathcal{F}_\sigma$), $\bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha = \mathcal{B}$ where ω_1 is the first uncountable ordinal. Each Borel set B (excluding the closed sets) occurs for the first time in some $\mathcal{B}_\beta$ and we call β the *order* of B.

If B has order $\beta > 0$, it is clear that β is not a limit ordinal, and thus either B is of a union type, i.e.

$$B = \bigcup_1^\infty B_n, \qquad B_n \in \mathcal{B}_{\beta-1}, \quad n = 1, 2, \cdots$$

or B is of intersection type, i.e.

$$B = \bigcap_1^\infty B_n, \qquad \mathcal{B}_n \in \mathcal{B}_{\beta-1}, \quad n = 1, 2, \cdots.$$

For each element $B \in \mathcal{B} \setminus \mathcal{B}_0$ we fix one such representation — either as a union or as an intersection of Borel sets of lower order. To implement his strategy, player II will construct an auxiliary sequence of finite families of sets $M_k = \{B_l^k : l < n_k\}$, $k = 0, 1, \cdots$ that will tell him how to choose the B_k's so that $\bigcap_1^\infty B_k \neq \emptyset$. We will first describe the properties that the M_k's will have and then explain how to inductively build them. The properties are:

(1) $B_k \subset B_l^k$ for all $1 \leqq l \leqq n_k$.

(2) $B_k \in M_k, \cdots \subset M_k \subset M_{k+1} \subset \cdots$.

(3) If B_l^k is of intersection type with representation $B_l^k = \bigcap_{i=1}^\infty B_{l,i}^k$ then

$$B_{l,i}^k \in M_{k+1} \qquad \text{for all } 1 \leqq i \leqq k.$$

(4) If B_l^k is of union type with representation

$$B_l^k = \bigcup_{i=1}^{\infty} B_{l,i}^k$$

then for some i, $B_{l,i}^k \in M_{k+1}$.

(5) M_k contains some closed interval I_k of length $\leqq 1/2^{k+1}$.

Now for the construction: To start with consider A_0, and note that either $A_0 \cap [0, \frac{1}{2}]$ is non-trivial or $A_0 \cap [\frac{1}{2}, 1]$ is non-trivial. Choose I_0 to be that interval so that $A_0 \cap I_0$ is non-trivial and set

$$M_0 = \{A_0 \cap I_0 = B_0, I_0\}.$$

The only relevant properties are (1), (2) and (5) and they are clearly satisfied. Suppose then that M_j has been constructed for $j \leqq k$ satisfying the properties (1)–(5). At stage $k+1$, player I selects some non-trivial $A_{k+1} \subset B_k$ and player II proceeds to construct M_{k+1} as follows:

He begins by putting all of M_k in M_{k+1} and then for each B_l^k of intersection type $\bigcap_1^\infty B_{l,i}^k$ he adds $B_{l,i}^k$, $1 \leqq i \leqq k$ to M_{k+1}. Next he takes the first (least l) B_l^k of union type (if any) and observes that $\bigcup_1^\infty B_{l,i}^k = B_l^k \supset B_k \supset A_{k+1}$ and thus, since $\mathscr{C}$ is a σ-ideal, there is some i such that $B_{l,i}^k \cap A_{k+1}$ is non-trivial. He adds one of these $B_{l,i}^k$'s to M_{k+i}, and denotes $E_1 = B_{l,i}^k \cap A_{k+1}$. He then takes up the next B_m^k of union type (if any), $\bigcup_1^\infty B_{m,j}^k$, and finds some j such that $B_{m,j}^k \cap E_1$ is non-trivial, adds $B_{m,j}^k$ to M_{k+1} and sets $E_2 = B_{m,j}^k \cap E_2$. He continues this L times in all, where L is the number of B_l^k's of union type. Finally, he divides I_k into two equal intervals, and denote by I_{k+1} that subinterval such that $E_L \cap I_{k+1}$ is non-trivial, add I_{k+1} to M_k, set $B_{k+1} = E_L \cap I_{k+1}$, and add B_{k+1} to M_k. Naturally M_{k+1} is taken to be M_k with all the additions described above.

In a straightforward manner one checks that the properties (1)–(5) continue to hold for M_{k+1}, and thus a strategy for player II has been completely specified. To see that this is a winning strategy we note that since $I_0 \supset I_1 \supset \cdots$ is a decreasing sequence of closed subintervals with length going to zero, $\bigcap_0^\infty I_k$ is a single point, say x. We will prove that $x = \bigcap_0^\infty B_k$ by proving that for all sets C in $\bigcup_0^\infty M_k = M$, $x \in C$. This last assertion will be proved by induction on the order of C. To begin with suppose that C is of order zero, i.e. a closed set, that occurs for the first time in M_k. Then by property (1), $C \supset B_k$ and hence $C \supset B_l$ for all $1 \geqq k$. But also $I_l \supset B_l$ for all $l \geqq k$ and thus the distance between $x \in I_l$ and C is at most the length of I_l. This tends to zero, and since C is closed this implies that $x \in C$. Now assume that for all $\alpha \leqq \beta$ if $C \in M$ is of order α, $x \in C$ and suppose that $C \in M$ is of order $\beta + 1$. If it is of intersection type, $C = \bigcap_1^\infty C_i$, by property (3), all the C_i's also belong to M, and since their order is $\leqq \beta$, $x \in C_i$ for all i and thus $x \in C$. If C is of union type, $\bigcup_1^\infty C_i$, then for some i, $C_i \in M$ by property (4), and the order of C_i is $\leqq \beta$ and thus $x \in C_i$ and *a fortiori* $x \in C$. This completes the induction and proves that $x \in C$ for all $C \in M$, in particular, by property (2) $B_k \in M$ for all k and thus $x \in \bigcap_0^\infty B_k$ as required.

We will use the proposition in the form of the following immediate corollary:

COROLLARY 2. *There is a function ϕ defined from finite non-increasing sequences of $\mathcal{B} \setminus \mathcal{E}$ to $\mathcal{B} \setminus \mathcal{E}$ such that if $C_1 \supset C_2 \supset \cdots$ is any non-increasing sequence then $\phi(C_1, C_2, \cdots, C_l) \subset C_l$, $l \geqq 1$, and if for all l, $C_{l+1} \subset \phi(C_1, C_2, \cdots, C_l)$ then $\bigcap_1^\infty C_l$ is non-empty and in fact consists of a single point.*

§3. The main construction

Throughout this section T will be an automorphism of (X, β) with no periodic points such that $X \notin \mathcal{W}$, the σ-ideal generated by the wandering sets. An easy consequence of the lack of periodic points is the following useful lemma:

LEMMA 3. *If C is any non-trivial set and $L \geqq 1$ is any positive integer there is a non-trivial set $D \subset C$ such that $D, T, D, \cdots, T^L D$ are pairwise disjoint.*

PROOF. Recall that X is the unit interval and denote by $d(x, y)$ the usual metric there. Observe that for each n the set

$$E_n = \{x : d(T^i x, T^j x) \geqq 1/n \text{ all } 0 \leqq i < j \leqq L\}$$

is in $\mathcal{B}$. Since T has no periodic points $\bigcup_1^\infty E_n = X$, and thus for some n_0, $E_{n_0} \cap C$ is non-trivial. Covering X with a finite number of sets F_i of diameter less than $1/2n_0$ we see that for some i_0, $F_{i_0} \cap E_{n_0} \cap C$ is non-trivial and it may be taken for D.

Combining this lemma with the proof of Poincaré's recurrence theorem gives

LEMMA 4. *If C is a non-trivial set then for some positive integer L there is a non-trivial set $D \subset C$ such that:*

(a) $T^j D \cap D = \emptyset$, $1 \leqq j \leqq L$;

(b) $T^L D \subset C$.

We now begin the construction of our Cantor set in X. We will use the following notation: $\Sigma = \bigcup_1^\infty \{0, 1\}^n$, for $\sigma \in \Sigma$; $|\sigma|$ will denote the length of σ, i.e., that n such that $\sigma \in \{0, 1\}^n$; and $\sigma\sigma'$ will denote the concatenation of $\sigma, \sigma' \in \Sigma$. By Lemma 3 there is some non-trivial A_0 such that $A_0 \cap TA_0 \neq \emptyset$. Set

$$B_0 = \phi(A_0), \qquad B_1 = \phi(TB_0).$$

Next apply Lemma 4 to find a non-trivial $A_{00} \subset T^{-1}B_1$ together with some $L_1 \geqq 1$ such that: $T^{L_1}A_{00} \subset T^{-1}B_1$ and $A_{00} \cap T^i A_{00} = \varnothing$ for $i < L_1$. We set now

$$B_{00} = \phi(A_0, A_{00}), \qquad B_{10} = \phi(B_1, TB_{00}),$$

$$B_{01} = \phi(A_0, T^{L_1-1}B_{10}), \qquad B_{11} = \phi(B_1, TB_{01}).$$

Once again we apply Lemma 4 to find a non-trivial $A_{000} \subset T^{-L_1-1}B_{11}$ together with some $L_2 \geqq 1$ such that: $T^{L_2}A_{000} \subset T^{-1}B_{11}$ and $A_{000} \cap T^i A_{000} = \varnothing$ for $i < L_2$. We set now

$$B_{000} = \phi(A_0, A_{00}, A_{000}), \qquad B_{100} = \phi(B_1, B_{10}, TB_{000}),$$

$$B_{010} = \phi(A_0, B_{01}, T^{L_1-1}B_{100}), \qquad B_{110} = \phi(B_1, B_{11}, TB_{010}),$$

$$B_{001} = \phi(A_0, B_{01}, T^{L_2-L_1-1}B_{110}), \qquad B_{101} = \phi(B_1, B_{10}, TB_{001}),$$

$$B_{011} = \phi(A_0, B_{01}, T^{L_1-1}B_{101}), \qquad B_{111} = \phi(B_1, B_{11}, TB_{011}).$$

The pattern should now be clear. We are constructing a family of sets $\{B\sigma : \sigma \in \Sigma\}$ and $A_0, A_{00}, \cdots, A_0^n$ that will have the following properties:

(1) If $\sigma_1^n, \sigma_2^n, \cdots, \sigma_{2^n}^n$ denote the elements of $\{0,1\}^n$ in lexicographic order from left to right, then there are integers l_j^n such that

$$B\sigma_{j+1} \subset T^{l_j^n} B\sigma_j, \qquad 1 \leqq j < 2^n$$

and for $i < l_j^n$, $T^i B\sigma_j$ is disjoint from $\bigcup_1^{2^n} B\sigma_j$;

(2) $A_{0^n} \subset T^{-M_{n-1}}B_{1^{n-1}}$, $M_{n-1} = \sum_{j=1}^{2^{n-1}-1} l_j^{n-1}$, $B_{0^n} = \phi(A_0, A_{00}, \cdots, A_{0^n})$,

$$B\sigma_{j+1}^n = \phi(B\sigma_{j+1}^n(1), B\sigma_{j+1}^n(1), \sigma_{j+1}^n(2), \cdots, T^1 B\sigma_j^n)$$

for $1 \leqq j < 2^n$.

If we have already done so for n, then $A_{0^{n+1}}$ is chosen by applying Lemma 4 to $T^{-M_n}B_{1^n}$, and finding $A_{0^{n+1}}$ and L_n so that $T^{L_n}A_{0^{n+1}} \subset T^{-M_n}B_{1^n}$ but $T^i A_{0^{n+1}} \cap A_{0^{n+1}} = \varnothing$ for $i < L_n$, for $1 \leqq j < 2^n$, we take for $l_j^{n+1} = l_j^n$, for $l_{2^n}^{n+1} = L_n - M_n$ (recall $M_n = \sum_1^{2^n-1} l_j^n$), and $l_{2^n+j}^{n+1} = l_j^n$ for $1 \leqq j < 2^n$. Then (2) for $n+1$ defines $B\sigma$ for all σ with length $n+1$.

Denote by $Y = \{0,1\}^{\mathbf{N}}$ and by $D : Y \to Y$ the mapping defined by considering elements of Y as 2-adic integers and setting $Dy = y + 1$. Corollary 2 applies to each sequence formed by $B_{y(1)}, B_{y(2)}, \cdots$ where $y \in Y$ and $y(n)$ represents the initial n-segment of y, and gives a point

$$u(y) = \bigcap_1^\infty B_{y(n)}.$$

The set of all these points, U, is of course in 1-1 correspondence with Y, and $\mathcal{B} \mid U$ coincides with the usual Borel structure on Y under this correspondence. What is crucial is that T_u, the transformation induced by T on U, corresponds to the odometric map, D, defined above. We formulate this conclusion as a proposition that summarizes the construction.

PROPOSITION 5. *There is a set $U \subset X$, and a one-to-one onto map $\theta : U \to Y = \{0,1\}^{\mathbb{N}}$ such that the θ is a Borel automorphism between the standard Borel structure on Y* and *$\mathcal{B} \mid U$, and such that $D\theta = \theta T_U$.*

Now there are plenty of ergodic non-atomic measures quasi-invariant for D on Y. Indeed, up to orbit equivalence they represent all possible behaviour (see, for example, [3]). Any such measure μ on U easily extends to an ergodic quasi-invariant measure for T on X by looking at $\bigcup_0^\infty T^n U$, and on

$$U_n = T^n U \setminus \bigcup_{j<n} T^j U$$

putting the measure $(1/2^n)T^n\mu \mid T^{-n}U_n$. But for the fact that the finite invariant measure on U can become infinite in this way, the orbit equivalence class doesn't change, and we have on X a representative of any type III, or the type II_∞ transformation. Had we been given a fixed non-trivial set A to begin with, the discussion could be carried out for $(A, \mathcal{B} \mid A, T_A)$ and thus we could have constructed an ergodic non-atomic quasi-invariant measure for T that gives positive measure to A. This proves the main theorem described in §1.

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